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GRAPH-THEORETICAL PROPERTIES OF MAJOR SUBMATRICES

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T. Kamae*

Abstract

Major submatrices of an nxm ($n \leq m$) matrix of rank n have an important role in the theory of a linear graph. This paper presents the graph-theoretical relationships among the major submatrices. "Adjacency" of two major submatrices is defined. Based on this new concept, a linear graph called a K-graph is defined, which represents the adjacencies among the major submatrices of a given matrix. The existence of a Hamilton circuit is shown in a K-graph.

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1. Introduction

Suppose we have an nxm ($n \leq m$) matrix M of rank n over a field F and that M does not contain any all-zero column. By a major submatrix (or, simply, a major) of M we mean a nonsingular submatrix of M of order n. Pick one major submatrix and call it a reference major M_0. Suppose that the columns of M_0 are the first n columns of M in the same order. Then we can write matrix M in the form $(M_0, \overline{M}_0)$. $\overline{M}_0$ is a submatrix of M consisting of all the columns of M which do not belong to M_0. If M_0 is the identity matrix of order n, matrix M is said to be in normal form. If M is a fundamental cutset matrix Q of a connected graph G, M can be put in normal form by the rearrangement of columns. It is well known that the major submatrices of Q are in one-to-one correspondence with the trees of G. (See [1], for example). W. K. Chen [2], S. L. Hakimi and N. Deo [3], and W. Mayeda [4] have recently found a method of obtaining all the major submatrices of M from a reference major M_0. As seen in their papers, all the trees of a connected graph G can be thus generated without duplication, since, in a fundamental cutset matrix of G, each column corresponds in one-to-one manner to an edge of G.

In this paper, graph-theoretical properties of majors of M are presented. First, the concept of adjacency between majors of M is introduced. Then a new graph (called the K-graph associated with matrix M) is defined, which represents the adjacencies of the majors. It is proved that there exists a Hamilton circuit in a K-graph. The proof can provide an efficient procedure for obtaining all the majors of matrix M. This paper is a generalized version of the author's previous work [5].

2. Definitions

Since M is an nxm ($n \leq m$) matrix of rank n, a major of M is an nxn matrix. In other words, each major M_i of M contains exactly n columns and all rows of M. Number the rows and columns of M from 1 to n or m, respectively. Note that M_0 consists of rows 1, 2, ---, n and columns 1, 2, ---, n. Two majors M_1 and M_2 are said to be adjacent if M_1 and M_2 share exactly n-1 columns. The K-graph associated with M is a linear graph in which the vertices are in one-to-one correspondence with the majors of M and the edges with the adjacencies between the corresponding majors.

Take, for example, a matrix

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

which is in normal form. The majors of this matrix are

$$M_0 = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_1 = \begin{array}{c} \begin{array}{ccc} 4 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_2 = \begin{array}{c} \begin{array}{ccc} 1 & 5 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_3 = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_4 = \begin{array}{c} \begin{array}{ccc} 4 & 5 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$M_5 = \begin{array}{c} \begin{array}{ccc} 1 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Among them, M_0 and M_1, M_0 and M_2, M_0 and M_3, M_1 and M_3, M_1 and M_4, M_2 and M_4, M_2 and M_5, M_3 and M_5, M_4 and M_5 are adjacent. Thus the K-graph associated with M is obtained as shown in Figure.

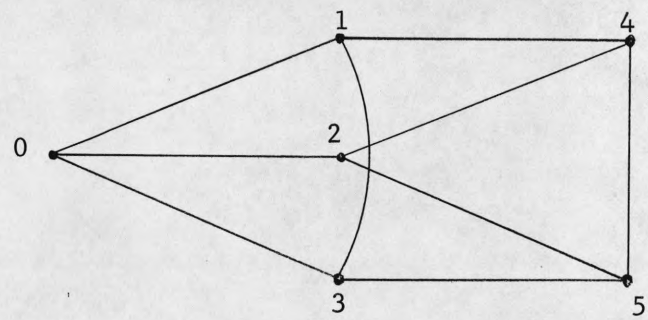

Figure A K-graph

By <u>section</u> $i_1 i_2 \text{---} i_k$, $1 \leq i_1 < i_2 <\text{---}< i_k \leq n$, <u>associated with M and M_0</u>, we mean the subgraph of a K-graph which consists of majors containing all columns of M_0 except columns i_1, i_2, ---, i_k, and of edges between the vertices corresponding to these majors. <u>Section 0</u> is a section consisting of only the vertex corresponding to M_0. If there is no such major, section $i_1 i_2 \text{---} i_k$ is said to be <u>empty</u>. It is evident that each major belongs to one and only one section, once a reference major M_0 is fixed. In the above example, M_2 belongs to section 2, M_4 to section 1·2, and sections 1·3 and 1·2·3 are empty.

Two nonempty sections S_i and S_j are said to be <u>adjacent</u> if there exist adjacent majors M_i and M_j such that $M_i \in S_i$ and $M_j \in S_j$. In the example, sections 2 and 1·2 are adjacent, since M_2 and M_4 are adjacent.

Section $i_1 i_2 --- i_k$ is said to be an <u>ancestor</u> (or a <u>descendant</u>) of section $j_1 j_2 --- j_h$ if $\{i_1, i_2, ---, i_k\} \subsetneq \{j_1, j_2, ---, j_h\}$ (or $\{i_1, i_2, ---, i_k\} \supsetneq \{j_1, j_2, ---, j_h\}$).

3. Properties of a Section Graph

Let A be a nonsingular matrix of order k over F. If each row of A is considered as a k-tuple vector, all the rows of A are linearly independent. Hence a set of these k vectors is a basis for the k-dimensional vector space over F an element of which is a k-tuple of F. This vector space is called the <u>row space</u> of matrix A. Let $v_1, v_2, \text{---}, v_k$ be those k-tuples such that v_i corresponds to row i of A, $1 \leq i \leq k$. Then any k-tuple of F is uniquely represented by a linear combination of $v_1, v_2, \text{---}, v_k$. Let

$$v_{k+1} = \alpha_1 v_{i_1} + \alpha_2 v_{i_2} + \ldots + \alpha_h v_{i_h} \qquad 1 \leq i_1 < i_2 < \ldots < i_h \leq k$$

$$\alpha_j (\neq 0) \in F, \; 1 \leq j \leq h$$

Matrix A is denoted as

$$A = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$$

<u>Lemma 1</u>

Let a_i be an element of F, $1 \leq i \leq k+1$. Let B be a square matrix of order k+1 represented as

$$\begin{bmatrix} v_1 & a_1 \\ v_2 & a_2 \\ \vdots & \vdots \\ v_k & a_k \\ v_{k+1} & a_{k+1} \end{bmatrix}$$

In other words, the ij-element of B is the j-th element of v_i if $1 \leq j \leq k$, and the i(k+1)-th element of B is a_i, $1 \leq i \leq k+1$. Then B is nonsingular if and only if

$$\alpha_1 a_{i_1} + \alpha_2 a_{i_2} + \ldots + \alpha_h a_{i_h} \neq a_{k+1}.$$

By the submatrix of M consisting of rows $i_1, i_2, ---, i_k$ and columns $j_1, j_2, ---, j_h$ we mean the kxh matrix whose st-element is the $i_s j_t$-element of M, $1 \leq s \leq k$, $1 \leq t \leq h$.

Elementary row operations for matrix M are defined to be:

1) interchange of any two rows of M.

2) multiplication of any row of M.

3) addition of any multiple of one row of M to another.

Suppose matrix N is obtained from matrix M by a succession of elementary row operations; in other words, M and N are row-equivalent. Then M and N have the same rank. According to the theory of matrix (refer to Hoffman and Kunze [6] for details), matrix M is row-equivalent to a row-reduced echelon matrix; that is, since M_0 is nonsingular, M = $(M_0, \overline{M}_0)$ is row-equivalent to matrix N = $(U, \overline{N}_0)$, where U is the identity matrix of order n, and $\overline{M}_0$ is row-equivalent to $\overline{N}_0$. Furthermore, if M_1 is a major of M consisting of all rows and columns $j_1, j_2, ---, j_n$, then the submatrix N_1 of N consisting of the same rows and the same columns is also a major of N. Elementary row operations do not affect the property of an nxn submatrix being a major. Hence we assume hereafter that M is in normal form.

The next lemma is due to Chen [2], Hakimi and Deo [3], and Mayeda [4].

Lemma 2

Suppose $1 \leq j_1 < j_2 < --- < j_k \leq n$ and $n+1 \leq j_{k+1} < j_{k+2} < --- < j_n \leq m$. A square submatrix M_i of M of order n which consists of all rows and columns $j_1, j_2, ---, j_k, j_{k+1}, j_{k+2}, ---, j_n$ is nonsingular (i.e., a major of M) if and only if the submatrix $M_i{}'$ of $\overline{M}_0$ of order n-k which consists of columns $j_{k+1}, j_{k+2}, ---, j_n$ and of rows $i_{k+1}, i_{k+2}, ---, i_n$ is nonsingular, where $\{i_{k+1}, i_{k+2}, ---, i_n\} = \{1, 2, ---, n\} - \{j_1, j_2, ---, j_k\}$. In other words, each major M_i of M except M_0 is in one-to-one correspondence with a nonsingular submatrix $M_i{}'$ of $\overline{M}_0$.

The following three lemmas provide information on the adjacency of nonempty sections.

Lemma 3

Let $0 \leq i_1 < i_2 < --- < i_{x-1} < i_x < i_{x+1} < --- < i_k \leq n$. Suppose sections $i_1 i_2 --- i_{x-1} i_{x+1} --- i_k$ (or S_1) and $i_1 i_2 --- i_{x-1} i_x i_{x+1} --- i_k$ (or S_2) are nonempty. For any major M_1 in S_1, there exists a major M_2 in S_2 which is adjacent to M_1, and vice versa. Consequently, section S_1 and section S_2 are adjacent.

Proof: Let $\{j_1, j_2, ---, j_{n-k}\} = \{1, 2, ---, n\} - \{i_1, i_2, ---, i_{x-1}, i_x, i_{x+1}, ---, i_k\}$. Then M_1 contains columns $j_1, j_2, ---, j_{n-k}$ and i_x, while a major in S_2 contains columns $j_1, j_2, ---, j_{n-k}$ but not column i_x. Let $j_{n-k+1}, j_{n-k+2}, ---, j_{n-1}$ be the other columns of M_1. Then the submatrix $M_1{}'$ of $\overline{M}_0$ consisting of rows $i_1, i_2, ---, i_{x-1}, i_{x+1}, ---, i_k$ and columns $j_{n-k+1}, j_{n-k+2}, ---, j_{n-1}$ of M is nonsingular by Lemma 2. The row space of $M_1{}'$ is generated from a basis $\{v_1, v_2, ---, v_{x-1}, v_{x+1}, ---, v_k\}$, where v_s is the (k-1)-tuple corresponding to row i_s of M restricted to M_1', $1 \le s \le k$ but $s \neq x$. Let v_x be the $(k-1)$-tuple corresponding to row i_x of M restricted to columns j_{n-k+1}, j_{n-k+2}, ---, j_{n-1}, and suppose

$$\sum_{j=1}^{k} \alpha_j v_j = 0$$

where $\alpha_j (\neq 0) \in F$, $1 \le j \le k$, and $\alpha_x = -1$. Since section S_2 is nonempty, there must exist a column j_n in $\overline{M}_0$ outside columns j_{n-k+1}, j_{n-k+2}, ---, j_{n-1} such that

$$\sum_{j=1}^{k} \alpha_j a_j \neq 0$$

where a_s is the $i_s j_n$-element of M, $1 \le s \le k$. Let

$$M_2' = \begin{bmatrix} v_1 & a_1 \\ v_2 & a_2 \\ \vdots & \vdots \\ v_{x-1} & a_{x-1} \\ v_x & a_x \\ v_{x+1} & a_{x+1} \\ \vdots & \vdots \\ v_k & a_k \end{bmatrix}$$

Then M_2' is nonsingular by Lemma 1. Let M_2 be a square submatrix of M of order n consisting of all rows and columns j_1, j_2, ---, j_{n-k}, j_{n-k+1}, ---, j_n. Then, by Lemma 2, M_2 is nonsingular, and hence a major submatrix of M in S_2. Obviously M_1 and M_2 are adjacent. For the converse, let M_2' be a nonsingular submatrix of $\overline{M}_0$ corresponding to M_2. M_2' contains rows i_1, i_2, ---, i_{x-1}, i_x, i_{x+1}, ---, i_k. Since M_2' is nonsingular, the cofactor of an element in row x of M_2' (which corresponds to row i_x of M) must be nonzero. Let M_1' be the matrix corresponding to such nonzero cofactor. Then major M_1 corresponding to M_1' belongs to section S_1 by Lemma 2 and is adjacent to M_2. This completes the proof.

Lemma 4

Let $1 \leq i_1 < i_2 < --- < i_{x-1} < i_x < i_{x+1} < --- < i_{y-1} < i_y < i_{y+1} < ---- < i_k \leq n$. Suppose that section $i_1 i_2 --- i_{x-1} i_x i_{x+1} --- i_{y-1} i_y i_{y+1} --- i_k$ (or S_0) is empty and that sections $i_1 i_2 --- i_{x-1} i_{x+1} --- i_{y-1} i_y i_{y+1} --- i_k$ (or S_1) and $i_1 i_2 --- i_{x-1} i_x i_{x+1} --- i_{y-1} i_{y+1} --- i_k$ (or S_2) are nonempty. For any major M_1 in S_1, there exists a major M_2 in S_2 which is adjacent to M_1, and vice versa. Consequently, sections S_1 and S_2 are adjacent.

Proof: Let $\{j_1, j_2, ---, j_{n-k}\} = \{1, 2, ---, n\} - \{i_1, i_2, ---, i_{x-1}, i_x, i_{x+1}, ---, i_{y-1}, i_y, i_{y+1}, ---, i_k\}$. Then M_1 contains columns $j_1, j_2, ---, j_{n-k}$ and i_x but not column i_y, while a major in S_2 contains columns $j_1, j_2, ---, j_{n-k}$ and i_y but not column i_x. Let $j_{n-k+1}, j_{n-k+2}, ---, j_{n-1}$ be the other columns of M contained in M_1. By Lemma 2, M_1', which is defined in the proof of Lemma 3, is nonsingular. Similarly to the previous proof, the row space of M_1' is generated from a basis $\{v_1, v_2, ---, v_{x-1}, v_{x+1}, ---, v_{y-1}, v_y, v_{y+1}, ---, v_k\}$. Let v_x be the (k-1)-tuple corresponding to row i_x of M but restricted to columns $j_{n-k+1}, j_{n-k+2}, ---, j_{n-1}$, which are contained in M_1'. Then v_x belongs to the row space of M_1'. Let

$$(2) \qquad \sum_{j=1}^{k} \alpha_j v_j = 0$$

where α's are elements in F and $\alpha_x = -1$. Suppose $\alpha_y = 0$. Since S_2 is nonempty, there must exist a column j_n in $\overline{M}_0$ outside columns j_{n-k+1}, --

---, j_{n-1} such that

$$\sum_{j=1}^{k} \alpha_j a_j \neq 0$$

where a_s is the $i_s j_n$-element of M, $1 \leq s \leq k$. Then, by Lemma 1, there exists a nonsingular submatrix of $\overline{M}_0$ of order k which contains rows i_1, i_2, ---, i_k. By Lemma 2, this contradicts the emptiness of section S_0. Hence, $\alpha_y \neq 0$; in other words, v_y must be included in the representation (2) of v_x as a linear combination of basis $\{v_1, v_2, ---, v_{x-1}, v_{x+1}, ---, v_{y-1}, v_y, v_{y+1}, ---, v_k\}$ of the row space of M_1'. Multiplying equation (2) by $-1/\alpha_y$, we have

$$v_y = \sum_{j=1}^{y-1} \beta_j v_j + \sum_{j=y+1}^{k} \beta_j v_j \qquad \beta_j = -\alpha_j/\alpha_y$$

Hence v_1, v_2, ---, v_{x-1}, v_x, v_{x+1}, ---, v_{y-1}, v_{y+1}, ---, v_k are linearly independent. Let

$$M_2' = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{x-1} \\ v_x \\ v_{x+1} \\ \vdots \\ v_{y-1} \\ v_{y+1} \\ \vdots \\ v_k \end{bmatrix}$$

From the above discussion, M_2' is a nonsingular submatrix of $\overline{M}_0$ of order k-1. By Lemma 2, we have a major M_2 corresponding to M_2', which is contained in S_2 and adjacent to M_1. By the duality we have the converse.

Lemma 5

All the descendants of an empty section are empty. Equivalently, all the ancestors of a nonempty section are nonempty.

Proof: If section $i_1 i_2 --- i_{x-1} i_x i_{x+1} --- i_k$ is nonempty, section $i_1 i_2 --- i_{x-1} i_{x+1} --- i_k$ is also nonempty by the proof of Lemma 3. By mathematical induction we have the conclusion.

The section graph associated with matrix M and reference major M_0 is defined to be a linear graph in which the vertices are in one-to-one correspondence with the nonempty sections associated with M and M_0 and the edges with the adjacencies of the sections provided by Lemmas 3 and 4. In the above example, the section graph associated with M and M_0 is identical with the K-graph, since all nonempty sections consist of only one major.

Lemmas 3, 4 and 5 lead to the following corollary.

Corollary

A section graph is connected.

4. A Hamilton Circuit in a K-graph

By a k-th section we mean a section having k distinct positive integers in its notation. Section 0 is called the 0-th section. Note that M is in normal form. Suppose at least one first section is nonempty. Then, by interchanging columns in M_0 and rows of M, we can make section 1 nonempty and the resulting matrix still in normal form. Hence we can assume without loss of generality that columns are arranged so that section 1 is nonempty whenever there exists a nonempty first section.

A Hamilton circuit (or path) is a circuit (or path) which contains all the vertices in a graph. In his recent work [5], the author proved the following theorem by using the lemmas similar to Lemmas 3, 4 and 5.

Theorem 1

There exists a Hamilton path between vertices corresponding to sections 0 and 1 in section graph SG, if it contains at least two vertices.

The outline of the proof is in the following. If all second sections of SG are empty, SG is a complete graph, and hence the theorem is obvious. If at least one second section of SG is nonempty, we can make section $1 \cdot 2$ nonempty by the renumbering of columns and rows of M (i.e., by interchanging columns in M_0 and rows of M). Consider the subgraph of SG which consists of the sections having 1 in their notation and of edges between them, and also the remaining part of SG. Both are isomorphic with a section graph containing vertices less than the original one. By Lemma 5, section 2 is nonempty, and by Lemma 3, there is an edge between section 2 and section $1 \cdot 2$. Mathematical induction on the number of vertices in SG completes the proof.

Consider section $i_1i_2\text{---}i_k$ (or S). All the majors belonging to this section contain columns j_1, j_2, ---, j_{n-k} of M, where $\{j_1, j_2, \text{---}, j_{n-k}\} = \{1, 2, \text{---}, n\} - \{i_1, i_2, \text{---}, i_k\}$. By Lemma 2, each major in S is in one-to-one correspondence with a major of the submatrix M' of $\overline{M}_0$ consisting of rows i_1, i_2, ---, i_k and all the columns contained in $\overline{M}_0$. If $k < n$, section S is the K-graph associated with matrix M', which contains less rows and less columns than matrix M. If $k = n$, section S is the K-graph associated with matrix $\overline{M}_0$. If matrix $\overline{M}_0$ does not contain any major, section $1\cdot 2\cdot\text{---}\cdot n$ is empty. Suppose matrix $\overline{M}_0$ contains some major. Put $\overline{M}_0$ in normal form by elementary row operations and denote the new matrix by $M^1 = (M_0^1, \overline{M}_0^1)$ where M_0^1 is the identity matrix of order n. We can form the section graph associated with M^1 and M_0^1. By repeating a finite number (e.g., p) of this manipulation we can reach the stage where the n-th section associated with M^p and M_0^p is empty.

Now we can state the main theorem of this paper, which can be proved in a similar way to the corresponding theorem in reference [5].

<u>Theorem 2</u>

There exists a Hamilton circuit in the K-graph K associated with a matrix M of rank n such that either $m \geq n+2$ excluding all-zero columns, or $n \geq 2$, $m = n + 1$ and column m contains at least two nonzeros.

The proof is outlined in the following. In the latter case, the theorem is evident. If $m \geq n + 2$, there exist at least three nonempty sections. Suppose that the theorem is true if $n \leq k-1$. Consider the case $n = k$. As discussed in the above, all sections except section $1\cdot 2\cdot\text{---}\cdot n$ are K-graphs associated with matrices with less rows and less columns than k. Hence they have a Hamilton circuit by inductive hypothesis. Therefore, if section 1·2·----·n is empty, K has a Hamilton circuit by Lemmas 3 and 4 and Theorem 1. From the above discussion we can prove inductively that section 1·2·---·n has a Hamilton circuit. Lemmas 3 and 4 and Theorem 1 lead to the conclusion. Furthermore, as in [5], we can prove that, for any two adjacent majors, there exists a Hamilton circuit in a K-graph with the same restrictions as in Theorem 2 such that the two majors are connected by an edge of that Hamilton circuit.

5. Concluding Remarks

The existence of a Hamilton circuit in a K-graph has been proved. As seen easily, the proof is also valid for an nxm ($n > m$) matrix of rank m. If a matrix M is a fundamental cutset matrix of a connected graph G, each major of M corresponds in one-to-one manner to a tree of G, and hence the K-graph associated with M is identical with the tree graph associated with G (see [5] for details). A tree graph is a special case of a K-graph. The proof of the existence (in Theorem 2) can be easily applied to obtaining all the majors of M along a Hamilton circuit of a K-graph.

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Major submatrices of an nxm)$n \leq m$) matrix of rank n have an important role in the theory of a linear graph. This paper presents the graph-theoretical relationships among the major submatrices. "Adjacency" of two major submatrices is defined. Based on this new concept, a linear graph called a K-graph is defined, which represents the adjacencies among the major submatrices of a given matrix. The existence of a Hamilton circuit is shown in a K-graph.

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